

A Method to Estimate the Primordial Power Spectrum from CMB Data

Rafael Bautista¹

Departamento de Física, Universidad de los Andes, A.A. 4976, Bogotá, Colombia

and

Sergio Torres²

Observatorio Astronómico, Universidad Nacional, A.A. 2584, Bogotá, Colombia
and, Centro Internacional de Física, A.A. 49490, Bogotá, Colombia

ABSTRACT

A precise determination of the primordial spectrum of matter density fluctuations at super-horizon scales is essential in understanding large scale structure in the universe. Attempts to constrain or obtain the primordial spectrum using data on cosmic microwave background (CMB) anisotropies has relied on statistical and correlation analyses that assume a power-law spectrum. We propose a method to derive $P(k)$ directly from the CMB angular power spectrum which does not presuppose the need to know anything about its functional form. The method consists of a direct inversion of the Sachs-Wolfe formula. Using this new analysis technique and COBE data we obtain an empirical $P(k)$ which 1) supports a power-law parameterization and 2) has an amplitude and spectral index consistent with previous analyses of the same data. We obtained for the spectral index, $n = 1.52 \pm 0.4$ when the 2nd year COBE data is used and $n = 1.22 \pm 0.3$ using the 4 year data set.

Subject headings: cosmic microwave background – large-scale structure of universe

1. Introduction

The mechanism for large scale structure formation in the universe calls for primordial density fluctuations (PDF) in the early universe. A knowledge of the spectrum of PDF, $P(k)$, would allow to compute the *rms* mass fluctuation on a given scale, $\delta M/M$ and the peculiar velocity field. Inflation predicts a scale invariant spectrum $P(k) \propto k$ (Harrison 1970, Zeldovich 1972).

¹e-mail: rbautist@cdnet.uniandes.edu.co

²e-mail: storres@cdnet.uniandes.edu.co

The availability of cosmic microwave background (CMB) data at large angle scales [COBE (Bennett et al. 1996, Bennett et al. 1994), TENERIFE (Hancock et al. 1994), and FIRS (Ganga et al. 1993)] has made it possible at least in principle to probe the shape of the PDF spectrum. Theoretical uncertainties (i.e. ‘cosmic variance’) and experimental constraints such as sampling variance, low signal-to-noise ratio and galactic contamination, however, impose very stringent limitations in the ability to obtain the original spectrum. In order to deal with the effects of an equatorial cut in the portion of the celestial sphere dominated by diffuse galactic emission, Górski et al. 1994 have found a new orthogonal set of basis functions to represent the scalar radiation field. An alternative used by Wright et al. 1994a (hereafter WRI94) applies weights to the spherical harmonic decomposition of $\Delta T/T$ in order to correct for aliasing among different ℓ -terms that result in the cut sphere when the monopole and dipole terms are removed.

Most of the analyses of CMB data aimed at probing $P(k)$, however, have been done using a statistical maximum likelihood analysis on the angular power spectrum or the auto-correlation function under the assumption of a power law for $P(k)$. In view of the above mentioned intrinsic and instrumental limitations, the need for new and alternative analysis methods is well justified. We propose a new technique to obtain $P(k)$ directly from the CMB angular power spectrum which *does not assume any particular form for $P(k)$* , thus allowing to test for deviations from power law models. The method is based on a direct integration of the Sachs-Wolfe formula (Sachs & Wolfe 1967) for the angular spectrum coefficients. It is shown that by a straightforward application of the mean-value theorem the Sachs-Wolfe integral can be inverted resulting in a robust estimate of $P(k)$ over the wave-length range available to CMB experiments.

2. The Algorithm

Expressing the CMB temperature anisotropies in the usual spherical harmonics expansion allows one to define the angular power spectrum, C_ℓ (Bond & Efstathiou 1987):

$$\frac{\Delta T}{T} = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\theta, \phi), \quad (1)$$

with

$$C_\ell \equiv \langle |a_{\ell m}|^2 \rangle. \quad (2)$$

The C_ℓ ’s used here are related to the rotationally invariant *rms* multipole moments used by COBE by $\Delta T_\ell^2 = (2\ell + 1)T_0^2 C_\ell / (4\pi)$, with T_0 the monopole temperature. Experiments provide estimates for C_ℓ and their connection with theory is established by means of the Sachs-Wolfe effect. For large angle scales the C_ℓ ’s are (Fabbri et al. 1987):

$$C_\ell = \left(\frac{4\pi}{5} \right)^2 \int_0^\infty P(k) \left(\frac{j_\ell(k)}{k} \right)^2 dk, \quad (3)$$

where we have made k adimensional ($k \leftarrow 2ck/H_0$) and the formula is valid for $\Omega_0 = 1$.

We attempt to get straightforward information about the spectrum of PDF by a direct inversion of equation (3). The primary data are the angular correlation coefficients measured by experiments. For this purpose we have devised a method which, in addition to its simplicity, has two basic virtues: 1) It makes a minimum of assumptions about the form of the spectrum and 2) it can be readily extended to include any new information on the C_ℓ 's that might come from future experiments.

Let us introduce the integral

$$I_\ell = \int_0^\infty \left(\frac{j_\ell(k)}{k} \right)^2 dk, \quad (4)$$

for $\ell \geq 2$.

Notice that every factor in the integrand of (3) is either positive definite or at least non-negative and smooth. This allows us to establish the following identity:

$$\begin{aligned} P(\bar{k}_\ell) &= \frac{1}{I_\ell} \int_0^\infty P(k) \left(\frac{j_\ell(k)}{k} \right)^2 dk \\ &= \frac{C_\ell}{\left(\frac{4\pi}{5} \right)^2 I_\ell} \end{aligned} \quad (5)$$

which is an application of the mean value theorem (Stromberg 1981). Equation (5) simply states that, for some value of its argument, here denoted as \bar{k}_ℓ , the value of $P(k)$ has to match the right-hand-side of (5). This is true under the conditions stated above. Then the evaluation of (4) and the knowledge of the C_ℓ 's for a given set of values of ℓ , yield the value of the left-hand-side of (5). In principle it can be seen that the Sachs-Wolfe formula combines all scales in k -space in the coefficients C_ℓ . The reason why the direct inversion prescription works can be seen by examining the form of the kernel, $K_\ell(k) \equiv (j_\ell(k)/k)^2$, in equation (5). The spherical Bessel functions are quasiperiodic with decreasing amplitude for large k , thus the kernel for each ℓ is also a periodic function but with its first peak being the only dominant contribution (see Fig. 1). The maximum of the peak is roughly located at $k \approx 0.7 + 1.1\ell$ (with the dimensionless k used here) thus for each ℓ in the kernel one is probing a well defined and independent scale.

Now we need to estimate with enough accuracy the arguments \bar{k}_ℓ , in order to complete the table $P(k)$ vs. k . A way in which this can be accomplished is by recycling, under a new light, Zeldovich's recourse of estimating functions through the use of power laws, whenever the range of values of k of physical interest is small (only very large scales in our case). Therefore, appealing to the same kind of reasoning, we proceed to obtain the values of k_ℓ by means the estimative relation

$$(\bar{k}_\ell)^s \approx \frac{1}{I_\ell} \int_0^\infty k^s \left(\frac{j_\ell(k)}{k} \right)^2 dk, \quad (6)$$

where s is any ‘reasonable’ value that must be within a certain range that satisfies the criteria of convergence and physical plausibility. Of course, the values \bar{k}_ℓ obtained in this fashion can not be very sensitive to the particular s chosen as estimator in formula (6). In fact, the use of k^s as an estimator function is not necessary, it is only one of the simplest that will do the job. Other, more elaborate, estimators could be used, but, as it will be seen, this is not necessary.

We have evaluated equation (6) for the range $0.5 \leq s \leq 1.5$ and $2 \leq \ell \leq 12$, with the results that are shown in Table 1. As it can be quickly noticed, the maximum effect caused by varying s occurs for small ℓ with a spread around $\pm 7\%$ in the worst case. For larger ℓ these variations grow smaller and their effect is not of importance.

3. Analysis and Conclusions

Using the Hauser-Peebles angular power estimator WRI94 obtain values for the T_ℓ^2 coefficients which are a linear combination of the C_ℓ ’s (see Table 1 in WRI94). A numerical integration of the inversion formula with $s = 1.0$ and the angular power spectrum in the range $3 \leq \ell \leq 18$ derived from WRI94 was used to obtain the $P(k)$ points illustrated in Fig. 2. The error bars come from the uncertainties in the COBE data for the C_ℓ ’s. Coefficients beyond $\ell = 18$ were not included because the angular power spectrum seen by COBE for those ℓ ’s is dominated by noise and the associated angular scales are beyond COBE’s angular resolution. The observed angular power spectrum C_ℓ^{obs} is related to the theoretical spectrum by $C_\ell^{obs} = G_\ell^2 C_\ell$, where G_ℓ is the beam profile in terms of the coefficients in an expansion in Legendre polynomials. We have used the G_ℓ ’s in Wright et al. 1994b. The C_ℓ ’s are obtained by inverting the $T_{\ell,\ell'}$ matrix in Table 1 of WRI94. This is a matrix of dimension $\ell_{\max} \times \infty$ which formally does not have an inverse. However, for the particular case under consideration an approximate inverse can be computed by noticing that the non-zero matrix elements away from the diagonal, follow a scaling law, $T_{ij} \propto |i - j|^{-2.12}$. The expansion of the C_ℓ ’s in terms of the T_ℓ ’s is truncated at a point where additional terms contribute a negligible amount relative to the measurement errors. Fig. 3 shows the resulting C_ℓ ’s.

One can test the power law ‘anzats’ by attempting to fit $P(k)$ to a function of the form $P(k) \propto Q^2 k^n$. Here Q denotes the *rms* quadrupole normalization, more commonly written as $Q_{\text{rms-ps}}$. A maximum likelihood method taking into account the full covariance matrix was used. To find the model dependent covariance matrix, $M(Q, n)$, a Monte Carlo procedure was followed: first, the model parameters n and Q are fixed to generate realizations of the CMB angular power spectrum. These realizations of C_ℓ coefficients follow a χ^2 distribution with $2\ell + 1$ degrees of freedom and have mean values given by formula (4.18) of Bond & Efstathiou (Bond & Efstathiou 1987). For each C_ℓ realization our inversion method delivers a corresponding $P(k)$. The average of these $P(k)$ ’s over the ensemble of N_r realizations, $\langle P(k) \rangle$, was computed and the covariance

matrix as well:

$$M_{ij}(Q, n) = \frac{1}{N_r - 1} \sum_m^{N_r} (P_m(k_i) - \langle P(k_i) \rangle) (P_m(k_j) - \langle P(k_j) \rangle). \quad (7)$$

Finally, the likelihood function $L(Q, n)$ was computed:

$$-2 \ln L(Q, n) = d^T M^{-1}(Q, n) d + \ln \det(M(Q, n)) + \text{const}, \quad (8)$$

where the deviation vector d is the difference of a data point and the corresponding theoretical mean value from Monte Carlo realizations, $d_i = P(k_i) - \langle P(k_i) \rangle$. The covariance matrix is normalized so that the second term of equation (8) equals the χ^2 statistic. This procedure was repeated for several values of Q and n forming a discrete sampling of $L(Q, n)$ inside a grid defined by the ranges n : $0.8 - 2.3$ in steps of $\Delta n = 0.05$ and Q : $4 - 28 \mu\text{K}$ in steps of $\Delta Q = 0.5 \mu\text{K}$. A much finer resolution in Q and n was later obtained by two dimensional interpolation of the above defined grid of $L(Q, n)$ points. It was verified that with 5000 realizations the results converged to a stable value.

The bias and the errors on the estimated model parameters were obtained using the Monte Carlo procedure described above but with input synthetic data (for a fixed model) with known Q_{in} and n_{in} . For each input realization one obtains a set of values $Q_{\text{max}}, n_{\text{max}}$ that maximizes the likelihood. The mean of the $Q_{\text{max}}, n_{\text{max}}$ points gives the bias and their dispersion gives the actual errors. The n parameter is biased upward by ≈ 0.03 and Q is biased in the opposite direction by ≈ 0.26 . The $1\text{-}\sigma$ errors are $\delta n = 0.2$ and $\delta Q = 3.0 \mu\text{K}$. This would give us the uncertainty due to ‘cosmic variance’ alone. The error on the parameters due to instrumental noise was estimated and added in quadrature. The latter contribution to the error was computed following the Monte Carlo procedure explained above but instead of generating the model dependent C_ℓ ’s we took one single realization of C_ℓ ’s (which was fixed throughout the procedure) and to it we added realizations of instrumental noise power spectrum. The noise coefficients $C_{\ell, \text{noise}}$ are directly obtained from the harmonic coefficients of $A - B$ map combinations. Since $A - B$ noise maps do not require galactic cut, a straightforward harmonic fit is applicable.

The debiased results for which $L(Q, n)$ is maximum are $n = 1.52 \pm 0.4$ and $Q = 16.3 \pm 6.0 \mu\text{K}$ which are consistent with WRI94. Fig. 3 shows the angular power spectrum corresponding to this best fit $P(k)$ and COBE’s data points. To give an idea of the goodness of fit, the χ^2/DOF at L_{max} is $30.612/14$.

We repeated the analysis with the C_ℓ coefficients ($3 \leq \ell \leq 18$) from the 4 year COBE data given by Tegmark (1996). For these data, the maximum likelihood analysis gives $n = 1.22 \pm 0.3$ and $Q = 16.3 \pm 4.5$. The analysis also reveals that these parameters are anticorrelated. That is, values of Q and n that follow the relation $Q(n) = 19.9 \exp[0.756(1 - n)]$ lay approximately inside the $2\text{-}\sigma$ contour level.

Our results for Q and n are consistent (within the error bars) with those obtained by the COBE group which are summarized in Table 4 of WRI94 and in Table 2 of Bennett et al. (Bennett

et al. 1996) and depending on the analysis method or the way the data was prepared (i.e. which map combination, exclusion or not of the quadrupole term, beam shape filter, etc) their results for n range from 1.02 ± 0.4 to $1.42^{+0.49}_{-0.55}$ for the 2 year results and from $1.23^{+0.23}_{-0.29}$ to $1.30^{+0.30}_{-0.34}$ for the 4 year results. One important fact worth noticing is that independent of the n values, a power law form for the spectrum of PDF is indeed consistent with the $P(k)$ obtained here directly from the CMB data without an *a priori* assumption about its shape.

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ℓ	$\bar{k}_\ell(s = 1.0)^a$	$\Delta\bar{k}_\ell(s = 1.5)$	$\Delta\bar{k}_\ell(s = 0.5)$
2	2.782	+0.18	−0.16
3	4.168	+0.20	−0.17
4	5.494	+0.23	−0.18
5	6.790	+0.26	−0.20
6	8.067	+0.28	−0.22
7	9.329	+0.30	−0.24
8	10.574	+0.31	−0.25
9	11.815	+0.33	−0.27
10	13.029	+0.33	−0.28
11	14.249	+0.34	−0.29
12	15.437	+0.34	−0.29

Table 1: Variation of estimated arguments \bar{k}_ℓ with s .

^aThe central value of the \bar{k}_ℓ ’s is taken for the $s = 1.0$ case. The last two columns are the deviations in \bar{k}_ℓ from the central value when $s = 1.5$ and 0.5 are used respectively

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Fig. 1.— Kernel function $K_\ell(k)$ for $\ell = 2$ (*solid*), $\ell = 9$ multiplied by 100 (*short dash*) and $\ell = 18$ multiplied by 1000 (*long dash*)

Fig. 2.— Spectrum of PDF as derived from COBE’s 2 yr angular power spectrum and best fit to a power-law function.

Fig. 3.— COBE 2 yr angular power spectrum (*points*) compared to the power spectrum corresponding to our best power-law fit of $P(k)$ in Fig. 2. The external lines define the $\pm 1\sigma$ band expected from cosmic variance.





